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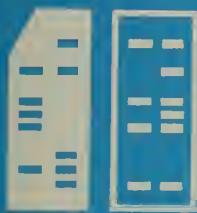
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DECOMPOSITION OF GRAPHS INTO TREES
by

S. Zaks
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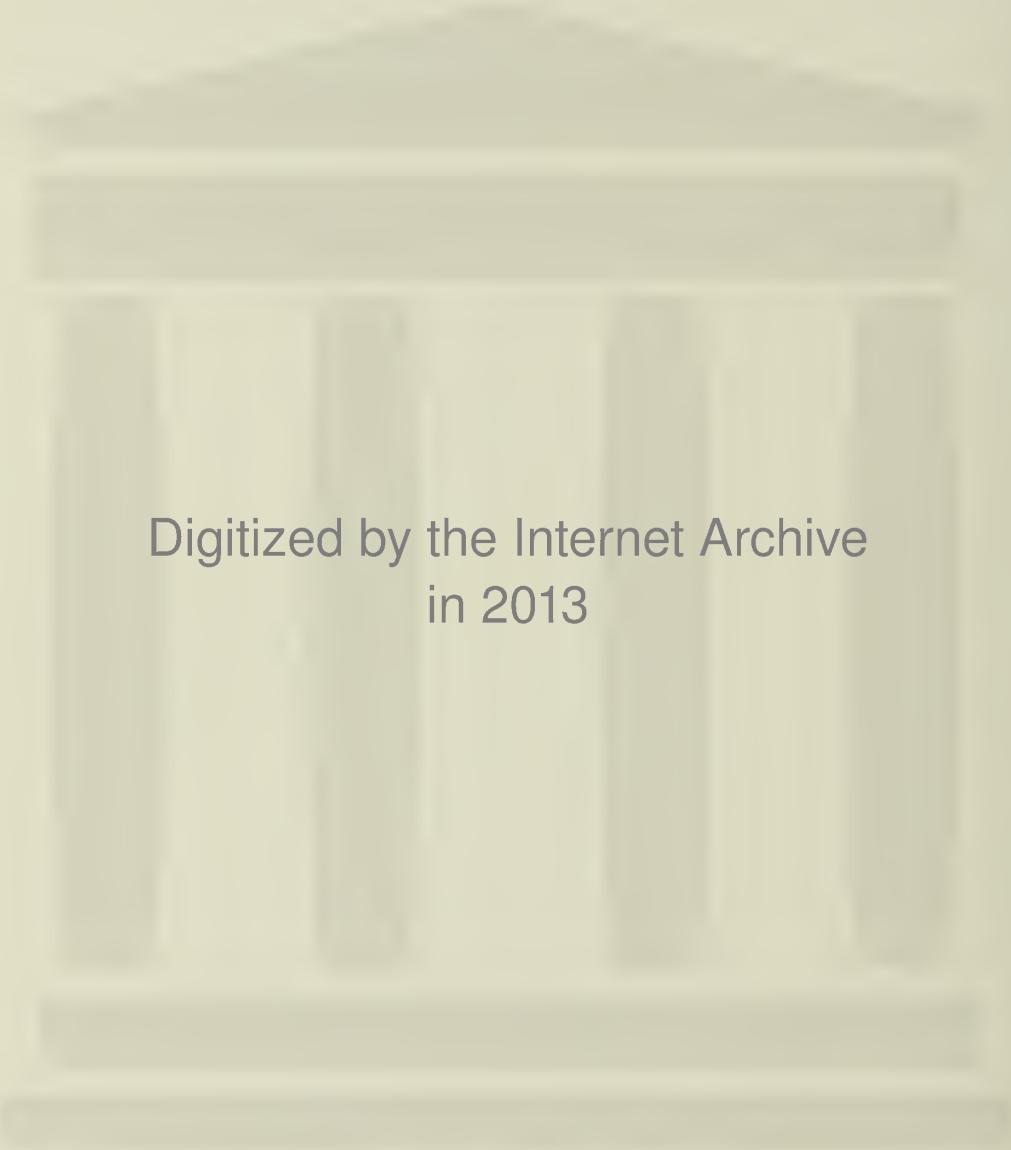


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DECOMPOSITION OF GRAPHS INTO TREES^{*}

by

S. Zaks and C. L. Liu

Department of Computer Science
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801

May 1977

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ABSTRACT

Denoting by T_i , P_i and S_i an i edges tree, path or star respectively, we show that the complete graph K_n can be decomposed into trees T_1, T_2, \dots, T_{n-1} , where T_i is either P_i or S_i for $i = 1, 2, \dots, n-1$. We also show that the same result holds for decomposing the complete bipartite graph $K_{n, \frac{n-1}{2}} (K_{\frac{n}{2}, n-1})$ for an odd (even) n . We then treat the cases of decomposing $K_{n,n}$ into $P_1, P_3, \dots, P_{2n-1}$ and $K_{n,n+1}$ (with an odd n) into P_2, P_4, \dots, P_{2n} . All of these results are being proved by decomposing the adjacency matrix of the appropriate graph. A different kind of result is shown for decomposing a full m -ary tree.

I. INTRODUCTION

In this paper we study the problem of decomposing a graph into edge-disjoint trees, where the graph is the complete graph K_n , or the complete bipartite graphs $K_{\frac{n}{2}, n-1}$ ($K_{\frac{n-1}{2}, n}$) for n even (odd) and $K_{n,n}$, or the full m -ary tree. We denote by T_i , P_i , S_i a tree, path, or star with i edges, respectively. We prove the following:

Theorem 1: K_n can be decomposed into T_1, T_2, \dots, T_{n-1} where T_i is either S_i or P_i for $i = 1, 2, \dots, n-1$.

Theorem 2: $K_{n,n}$ can be decomposed into $P_1, P_3, \dots, P_{2n-1}$.

Theorem 3: $K_{n,n+1}$ can be decomposed into P_2, P_4, \dots, P_{2n} for odd n .

Theorem 4: $K_{\frac{n}{2}, n-1}$ ($K_{\frac{n-1}{2}, n}$) for n even (odd) can be decomposed into T_1, T_2, \dots, T_{n-1} where T_i is either S_i or P_i for $i = 1, 2, \dots, n-1$.

Theorem 5: The full m -ary tree with k levels can be decomposed into $T_m, T_{m^2}, \dots, T_{m^k}$ in a unique way (up to isomorphism).

A. Gyafra and J. Lehel discuss in [1] this problem, referring to it as packing trees into K_n , and prove that T_1, \dots, T_{n-1} can be packed into K_n if all but two of them are stars, and also prove Theorem 1 by an induction procedure on n . J. F. Fink and H. J. Straight show in [2] that K_n can be decomposed into P_1, P_2, \dots, P_{n-1} (a special case of Theorem 1), that $K_{\frac{n}{2}, n-1}$ ($K_{\frac{n-1}{2}, n}$) for n even (odd) can be decomposed into P_1, P_2, \dots, P_{n-1} (a special case of Theorem 4), and have the same results as Theorem 2 and Theorem 3. They also show that $K_{n,n+1}$ can be decomposed into $P_2, P_4, \dots, P_{2n-2}$ and C_{2n} , a cycle of length $2n$, for an even n . They conjecture that Theorem 3 does not hold for an even n .

II. SOME BASIC OBSERVATIONS

We first show several compact ways to represent stars and certain paths in the adjacency matrix of a graph. Unfortunately, such representations can not be extended immediately to include all trees - even not all paths - which makes the problem of decomposing a graph to an arbitrary set of trees much more difficult.

Let G be an undirected graph - without loops and multiple edges - with vertices set $V = \{v_1, v_2, \dots, v_n\}$ and edges set E . Denote by $A(G) = (a_{ij})$ the adjacency matrix of G , such that $a_{ij} = 1$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$ otherwise. Note that $a_{ii} = 0$ for all i . We can use only the upper right part of $A(G)$ - (a_{ij}) for $1 \leq i < j \leq n$ - because G is undirected, and we denote it by $A^R(G)$.

In $A^R(G)$ name each of the following sequences of 1's as a stair:

(a) A right stair: $a_{i,j}, a_{i,j+1}, a_{i-1,j+1}, a_{i-1,j+2}, \dots, a_{i-\ell,j+\ell}, a_{i-\ell,j+\ell+1}$

(b) A left stair: $a_{i,j}, a_{i-1,j}, a_{i-1,j-1}, a_{i-2,j-1}, \dots, a_{i-\ell,j-\ell}, a_{i-\ell-1,j-\ell}$

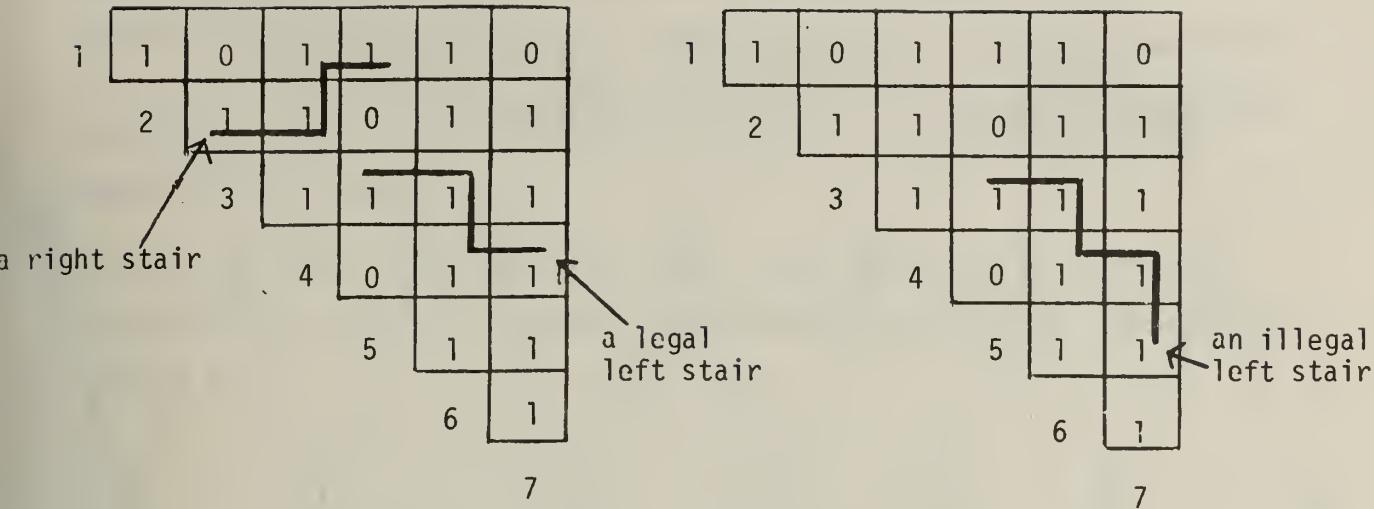
where for both (a) and (b) $\ell \geq 0$, all the a 's in the sequence equal 1, and each of the first and last elements can be excluded.

A left stair is legal if it doesn't contain both $a_{i,i+\ell}$ and $a_{i-\ell,i}$ for any i and ℓ , and is illegal otherwise.

For example, see the heavy lines in Figure 1, in which i stands for $v_i, i = 1, 2, \dots, 7$. (We use this notation in all the figures.)

Lemma 1: A star subgraph in a graph G corresponds to a set of 1 entries, all of them in one row or one column, in $A(G)$, and vice versa.

Proof: Immediate. \square



STAIRS IN THE ADJACENCY MATRIX

Figure 1

Lemma 2: 1. A right stair in $A^R(G)$ corresponds to a path in a graph G .
 2. A left stair in $A^R(G)$ corresponds to a path in a graph G iff it is legal.

Proof: Every stair in $A(G)$ corresponds to distinct consecutive edges in G , so it only remains to show that no cycle is formed.

In (1) we begin with a_{ij} , which corresponds to the edge (v_i, v_j) for $i < j$, and alternately increase the second index and decrease the first one, so each index k appears in at most 2 consecutive terms, and thus in G no cycle is formed by the appropriate edges. In (2) we begin with a_{ij} , which corresponds to the edge (v_i, v_j) for $i < j$, and alternately decrease the first and the second indices, so a cycle could be formed in the appropriate edges in G iff we have in the stair both $a_{i,i+\ell}$ and $a_{i-\ell,i}$

for some i and ℓ , hence the left stair in $A^R(G)$ corresponds to a path in G iff it is legal. \square

For example, the paths corresponding to the right and the legal left stairs, and the cycle corresponding to the illegal left stair of Figure 1, are shown in Figures 2.1, 2.2, and 2.3, respectively.

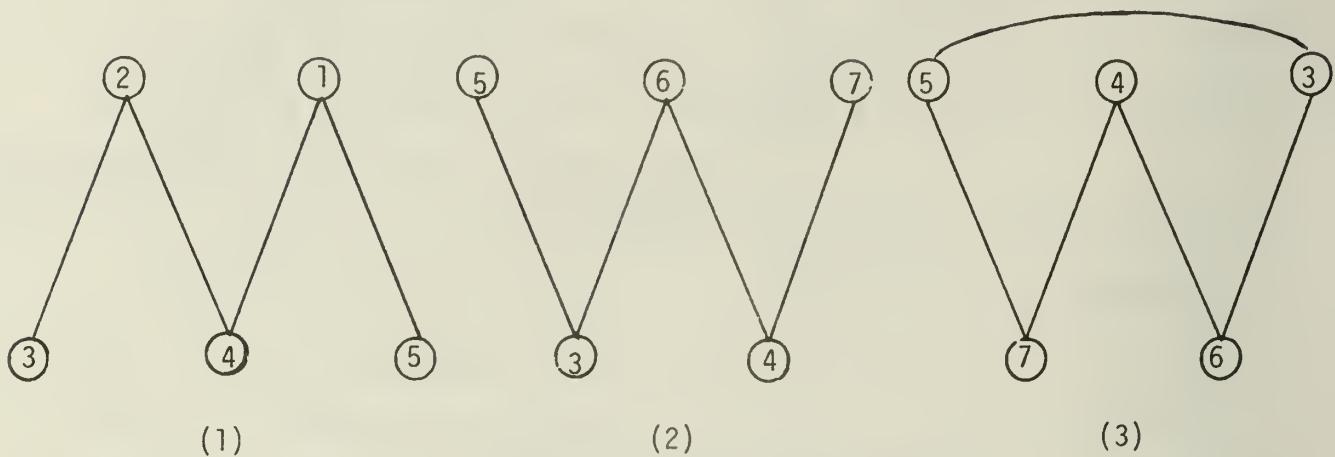


Figure 2

It should be noted that although the notion of left stair is interesting by itself, in our decompositions we will use only the right stairs.

III. DECOMPOSITION OF K_n

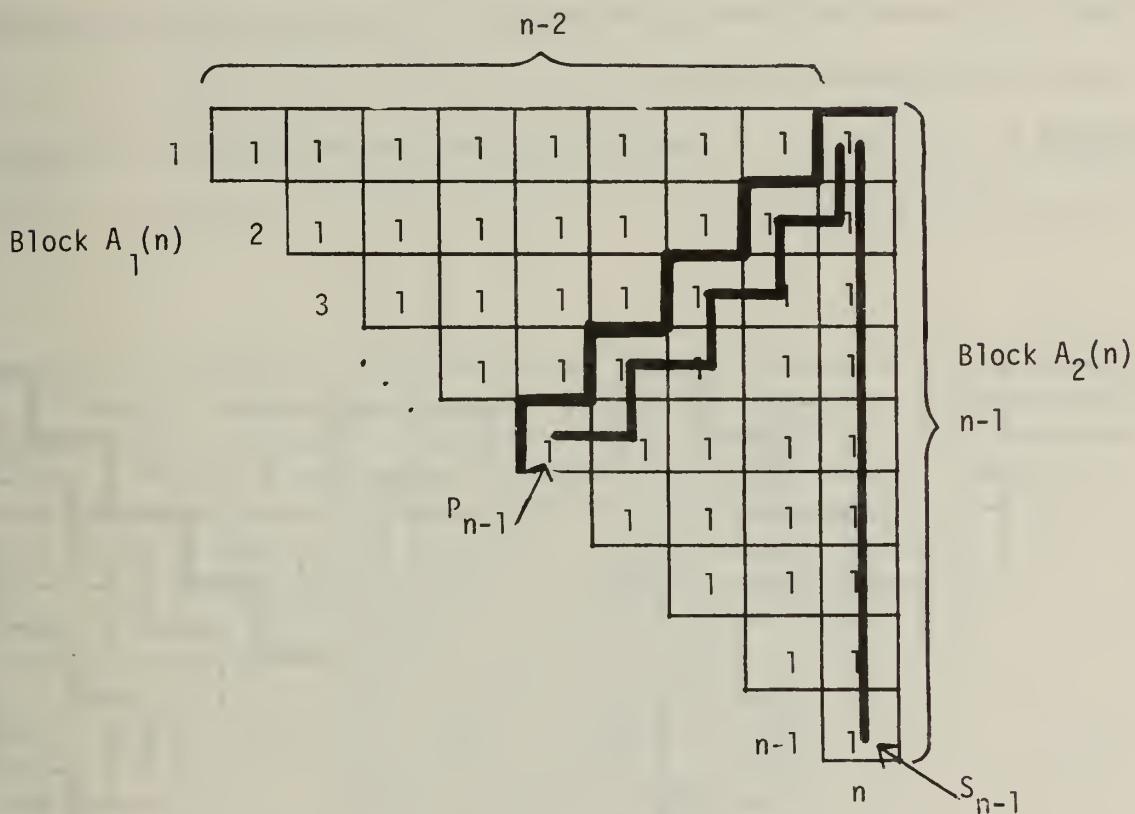
Recall that in $A^R(K_n)$ $a_{ij} = 1$ for all $1 \leq i < j \leq n$.

Theorem 1: K_n can be decomposed into T_1, T_2, \dots, T_{n-1} where T_i is either S_i or P_i for $i = 1, 2, \dots, n-1$.

Proof: We prove the theorem for even n . When n is odd, the argument is similar and is left to the reader. We divide $A^R(K_n)$ into two blocks (See Figure 3):

$$A_1(n) = \{a_{ij} \mid 1 \leq i < j \leq n, i + j \leq n\}$$

$$A_2(n) = \{a_{ij} \mid 1 \leq i < j \leq n, i + j > n\}$$



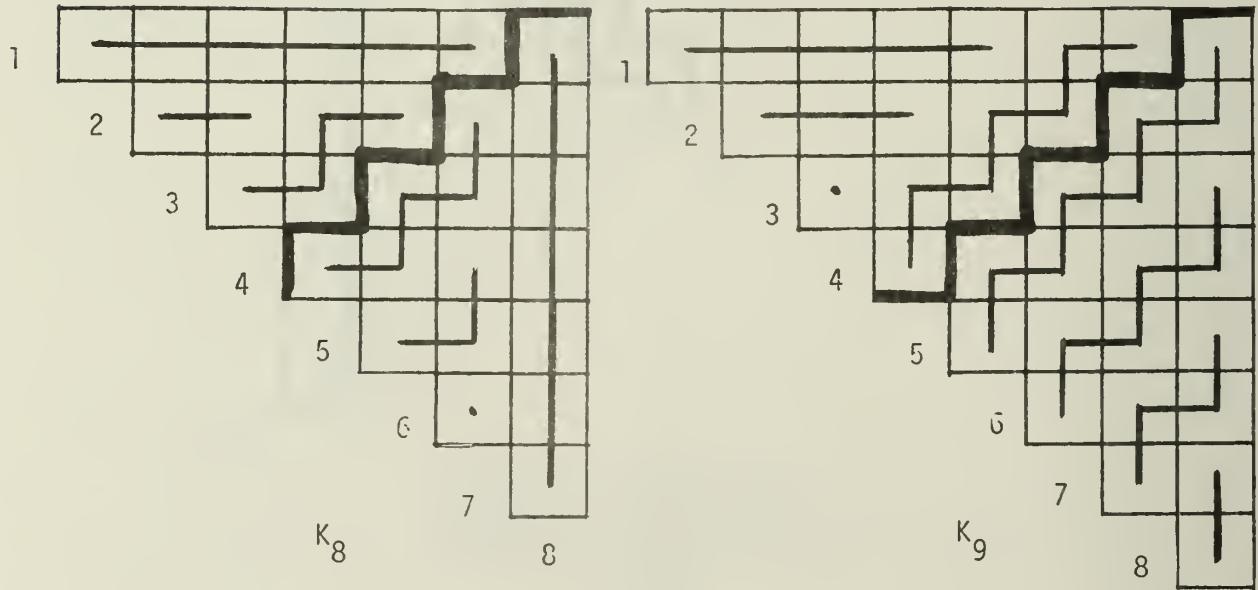
DECOMPOSITION OF $A^R(K_n)$

Figure 3

We now show that $A_1(n)$ can be decomposed corresponding to the given trees T_i for even i , and that $A_2(n)$ can be decomposed corresponding to the given trees T_i for odd i . The proofs for $A_1(n)$ and $A_2(n)$ are similar, so let us show the result for $A_2(n)$, by induction:

For $n = 1$ $A_2(n)$ is reduced to a 1×1 array, for which it is clear that P_1 or S_1 can be packed in. Assume it holds for an even $m < n$. Then for n we want to pack T_1, T_3, \dots, T_{n-1} into $A_2(n)$, when each T_i is either a path or a star. We consider two cases of having $T_{n-1} = P_{n-1}$ or $T_{n-1} = S_{n-1}$. Omitting either of them from $A_2(n)$ we are left with a block of $A_2(n-1)$'s shape as is shown in Figure 3, and in which we can pack all the rest of the trees by the induction hypothesis. \square

Example 1: In Figure 4 we show the decomposition of K_8 into $S_7, S_6, P_5, P_4, P_3, P_2, P_1$ and of K_9 into $P_8, P_7, P_6, S_5, P_4, S_3, S_2, S_1$



DECOMPOSITION OF K_n

Figure 4

IV. DECOMPOSITION OF BIPARTITE GRAPHS

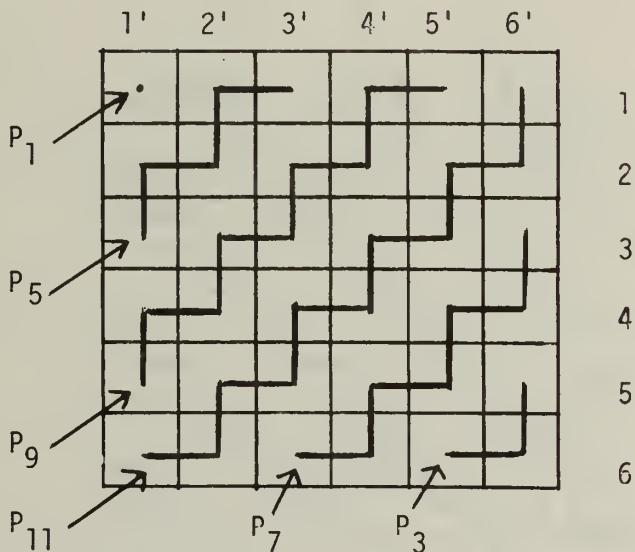
Recall that in $A^R(K_{p,q})$ we have a full $p \times q$ rectangular array of 1's, with 0's outside. In our discussion we work only with this rectangle.

Theorem 2: $K_{n,n}$ can be decomposed into $P_1, P_3, \dots, P_{2n-1}$.

Proof: The right stair that begins in the lower leftmost 1 and ends in the upper rightmost 1 corresponds to a path P_{2n-1} of length $2n-1$ in $K_{n,n}$.

After omitting it we are left with two separated parts of the $n \times n$ square, but as we are interested only in right stairs here, we can move those parts toward each other getting an $(n-1) \times (n-1)$ square, and the proof is thus completed by induction. \square

Example 2: The decomposition of $K_{6,6}$ into P_1, P_3, P_5, P_7, P_9 and P_{11} is shown in Figure 5.



DECOMPOSITION OF $K_{n,n}$

Figure 5

Theorem 3: $K_{n,n+1}$ can be decomposed into P_2, P_4, \dots, P_{2n} for odd n .

Proof: According to Theorem 2 we can decompose $K_{n,n}$ into $P_1, P_3, \dots, P_{2n-1}$. From this decomposition we can easily get the desired decomposition as illustrated in Figure 6. \square

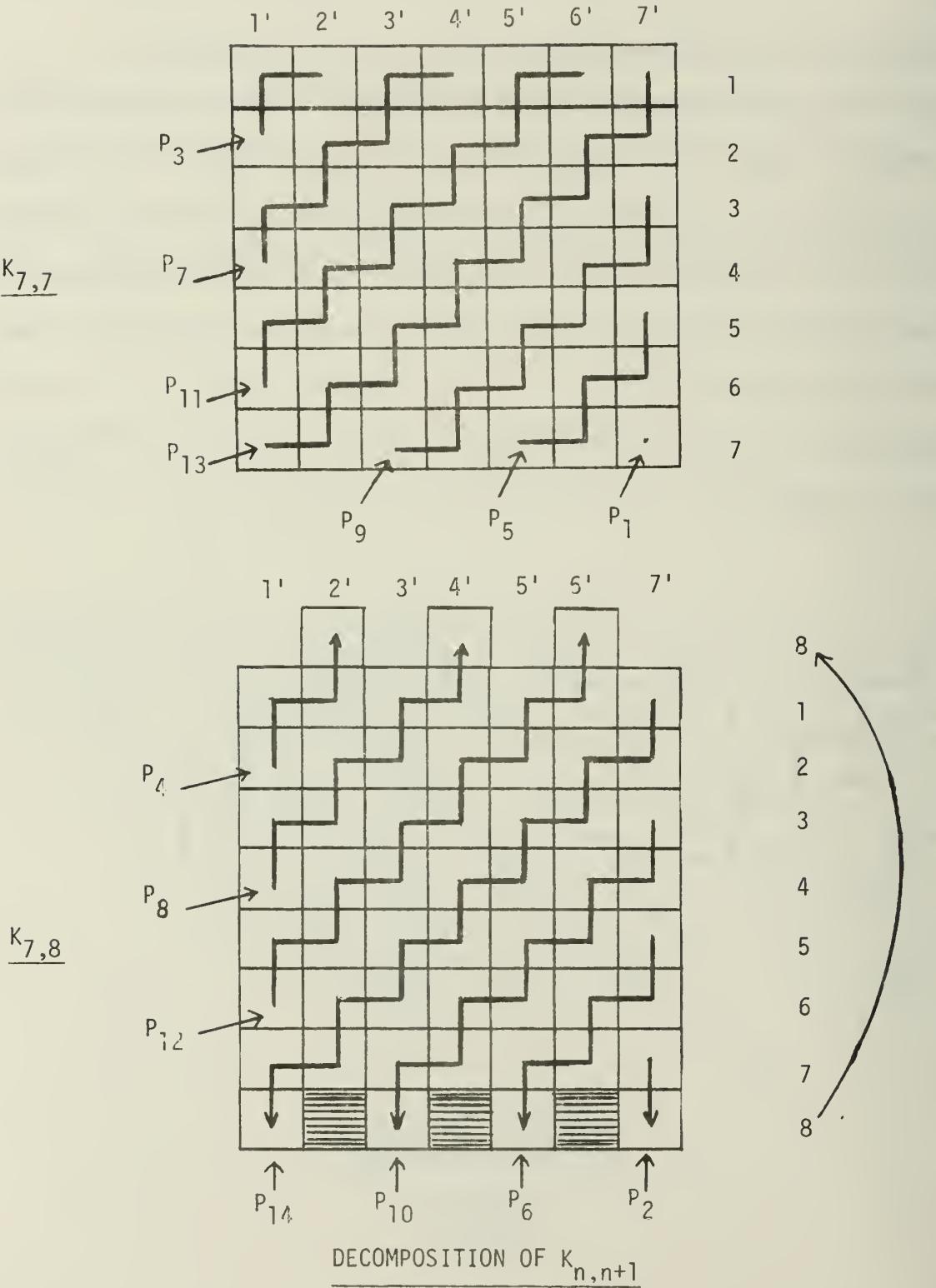
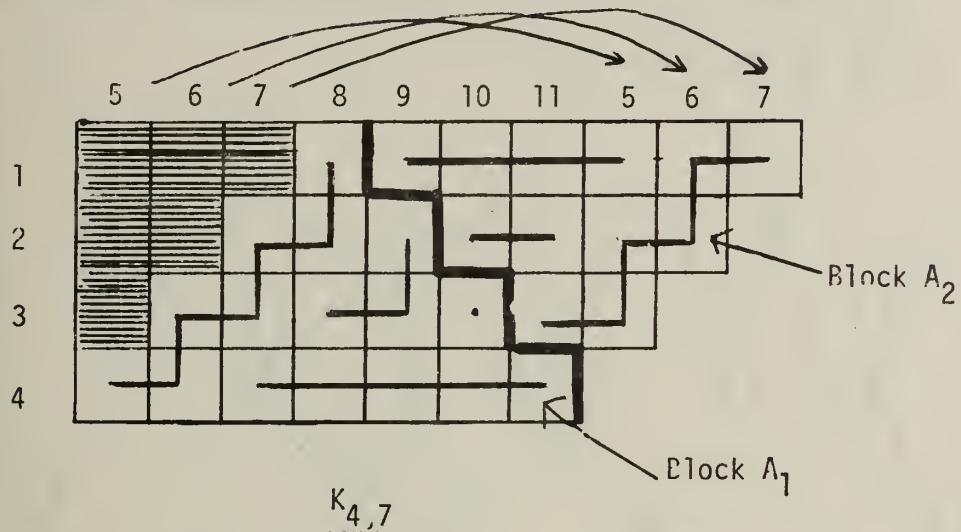


Figure 6

Theorem 4: $K_{\frac{n}{2}, n-1}$ ($K_{\frac{n-1}{2}, n}$) for n even (odd) can be decomposed into

T_1, T_2, \dots, T_{n-1} where T_i is either S_i or P_i for $i = 1, 2, \dots, n-1$.

Proof: The proof is similar to that of Theorem 1, so we only illustrate it here in an example where $n = 8$, and we decompose $K_{4,7}$ into $P_7, P_6, S_5, S_4, P_3, S_2, P_1$ (see Figure 7). \square



DECOMPOSITION OF $K_{\frac{n}{2}, n-1}$

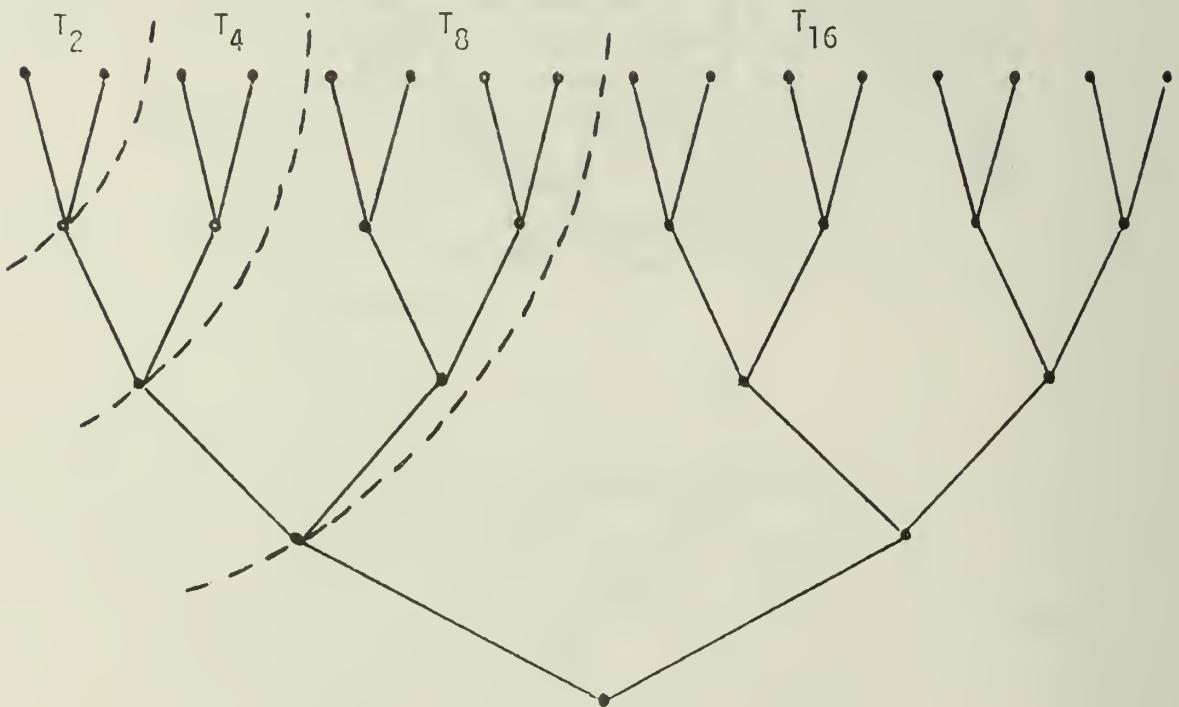
Figure 7

V. DECOMPOSITION OF FULL TREES

The last result about decomposition of graphs is obtained for a full tree, as follows:

Theorem 5: The full m -ary tree with k levels can be decomposed into $T_m, T_{m^2}, \dots, T_{m^k}$ in a unique way (up to isomorphism).

Example: The decomposition of the full binary tree with 4 levels into T_2, T_4, T_8, T_{16} is shown in Figure 8.

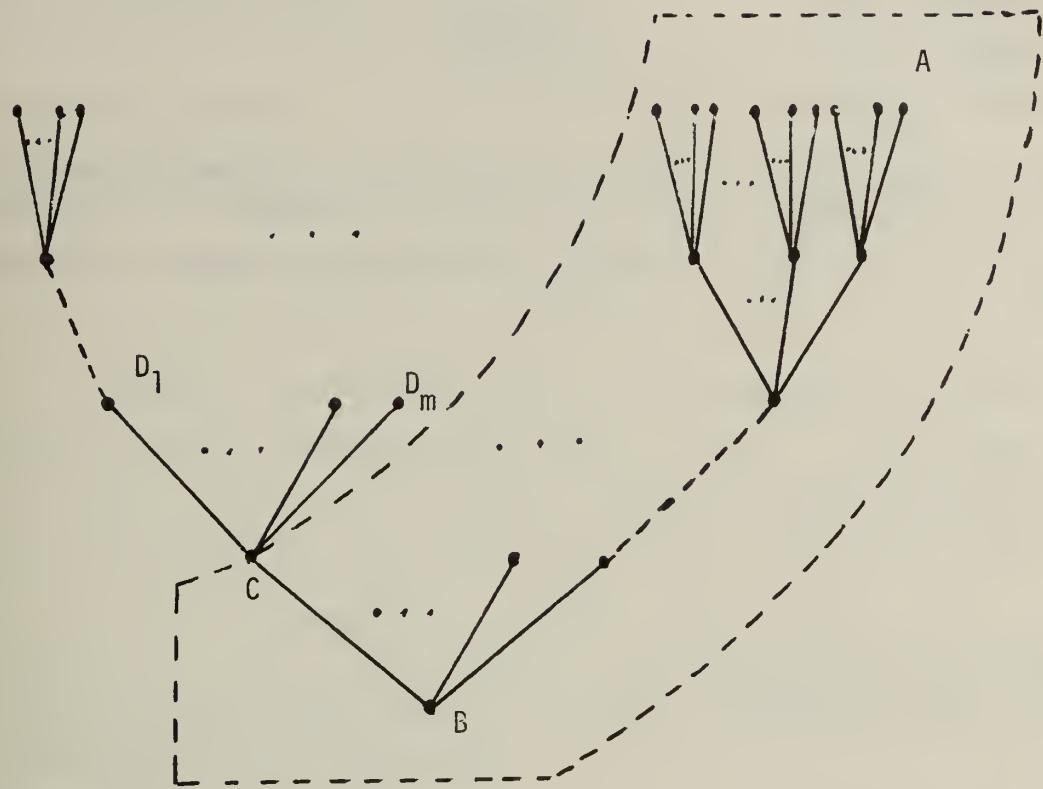


DECOMPOSITION OF A FULL BINARY TREE

Figure 8

Proof: Since a full m -ary tree of k levels has $\frac{m(m^k-1)}{m-1}$ edges and $\frac{m^k-1}{m-1}$ internal nodes, therefore T_{m^k} must contain at least one leaf, say A (see Figure 9), and the root B, as a simple counting argument shows. Thus all the path joining A and B belongs to T_{m^k} .

If any of the edges CD_1, \dots, CD_m belongs to T_m^k , we don't have enough room for T_{m-1}^k , as a simple counting



DECOMPOSITION OF A FULL M-ARY TREE

Figure 9

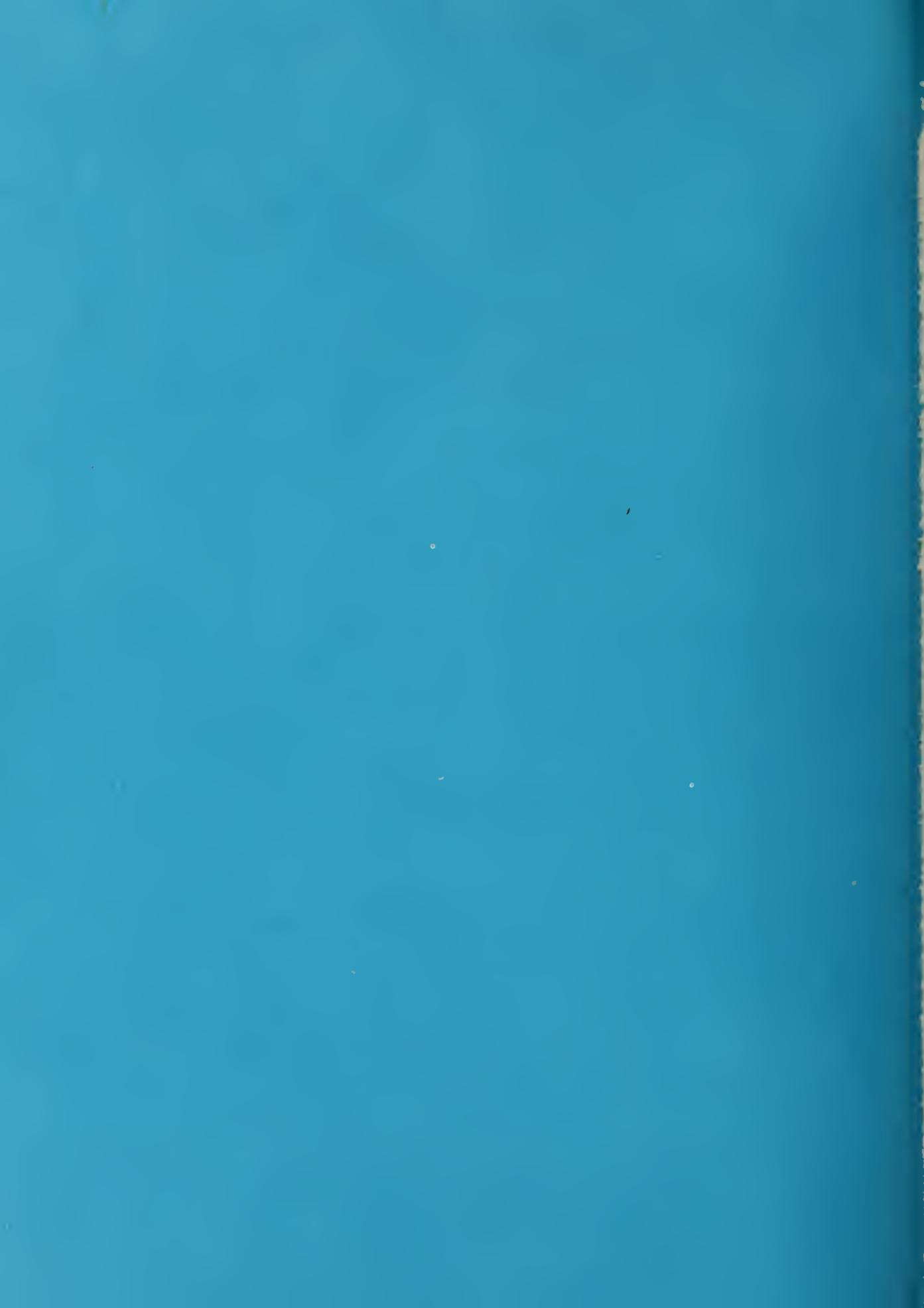
argument shows. So T_m^k must be in the enclosed area, where there are exactly m^k edges, and the rest of the proof follows by induction. \square

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- [1] A. Gyárfás and J. Lehel, "Packing Trees of Different Order Into K_n ", Proceedings of the 1976 Kenthely Colloquium, to be published.
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